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LINEAR FUNCTIONALS OF N-SPREADS.*

By Charles Albert Fischer.

It has been proved by Riesz that a linear functional of a curve, which is continuous with zeroth order, can be expressed as a Stieltjes integral.† The multiple Stieltjes integral has been defined by Fréchet and proved to be a linear functional.‡ In the first two sections of the present paper some properties of functions with finite variation, and the Stieltjes integrals determined by them, are discussed. In the last section Riesz' work is extended to functionals of n-spreads, and it is proved that when the functional is put in the form

$$A[f(x_1, \cdots, x_n)] = \int_T f(x_1, \cdots, x_n) d_n \alpha(x_1, \cdots, x_n),$$

the function $\alpha(x_1, \cdots, x_n)$ is unique if it is required to be regular. This makes it possible to extend Fréchet's work on bilinear functionals§ to functionals of n-spreads.

§ 1. **On functions with finite variation.** The variation $V_n\alpha(x)$ of a function $\alpha(x_1, \cdots, x_n)$ in the region

$$T: \qquad a_r \leqq x_r \leqq b_r \qquad (r = 1, 2, \cdots, n),$$

has been defined as the least upper bound of the expression‖

$$\sum_{i_1, \ldots, i_n} \left| \overset{(n)}{\underset{i_1, \ldots, i_n}{\Delta}} \alpha(x_1, \cdots, x_n) \right|,$$

where

$$\overset{(n)}{\underset{i_1, \ldots, i_n}{\Delta}} \alpha(x_1, \cdots, x_n) = \overset{(n-1)}{\underset{i_1, \ldots, i_{n-1}}{\Delta}} [\alpha(x_1, \cdots, x_{n-1}, x^{(i_n+1)}) - \alpha(x_1, \cdots, x^{(i_n)})],$$

and

$$\overset{(1)}{\underset{i_1}{\Delta}} \alpha(x_1, \cdots, x_n) = \alpha(x^{(i_1+1)}, x_2, \cdots, x_n) - \alpha(x^{(i_1)}, \cdots, x_n).$$

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* Read before the American Mathematical Society, December 27, 1916.

† Riesz, Annales Scientifiques de l'École Normale Supérieure, Ser. 3, Tome 31 (1914), pp. 9–14.

‡ Fréchet, Nouvelles Annales de Mathématiques, Ser. 4, Tome 10 (1910), pp. 247 and 254.

§ Fréchet, Transactions of the American Mathematical Society, vol. 16 (1915), pp. 215–34.

‖ Fréchet, loc. cit. (Nouvelles Annales), p. 242.

The multiple Stieltjes integral is defined by the equation*

$$\text{(1)} \qquad \int_T f(x)d_n\alpha(x) = \operatorname*{limit}_{\max(x^{(i_r+1)}-x^{(i_r)})=0} \sum_i f(\xi^{(i)}) \overset{(n)}{\underset{i}{\Delta}} \alpha(x),$$

where the arguments x, i and $\xi^{(i)}$ represent $x_1, \cdots, x_n$, $i_1, \cdots, i_n$ and $\xi^{(i_1)}, \cdots, \xi^{(i_n)}$ respectively, and $(\xi^{(i_1)}, \cdots, \xi^{(i_n)})$ is a point in the region $x^{(i_r)} \leqq x_r \leqq x^{(i_r+1)}$. If $V_n\alpha(x)$ is finite the integral exists for every continuous function $f(x)$. The function $\alpha(x)$ can always be replaced by a function which satisfies the equations

$$\alpha(x)\,|_{x_r=a_r} = 0, \qquad (r = 1, 2, \cdots, n),$$

without affecting its variation or the integral (1).† It will be assumed for convenience that this is always done. It follows that when r of the variables are given arbitrary values, the variation of $\alpha(x)$ considered as a function of the other $n - r$ variables cannot be greater than $V_n\alpha(x)$, and similarly $|\alpha(x)| \leqq V_n\alpha(x)$.

An $(n - 1)$-flat $x_s = \xi_s$ will be called singular when the variation of $\alpha(x)$ in the region $(\xi_s - \delta \leqq x_s \leqq \xi_s + \delta;\ a_r \leqq x_r \leqq b_r,\ r \neq s)$ does not approach zero with δ.‡ It will now be proved that a necessary and sufficient condition that $x_s = \xi_s$ be singular is that $\alpha(x)$ be discontinuous in x_s at some point of T where $x_s = \xi_s$. If it is singular the variation of $\alpha(x)$ in one of the regions $(\xi_s - \delta \leqq x_s \leqq \xi_s)$ or $(\xi_s \leqq x_s \leqq \xi_s + \delta)$ must be greater than some $k > 0$ which is independent of δ. For convenience this will be assumed for the last of these regions. The whole of T can be subdivided in such a way that the inequality

$$\text{(2)} \qquad V_n\alpha(x) - \sum_i \left| \overset{(n)}{\underset{i}{\Delta}} \alpha(x) \right| < \frac{k}{2}$$

is satisfied. It will not affect this inequality to assume that some $x^{(i_s)} = \xi_s$. Then δ can be taken as an arbitrary constant less than $x^{(i_s+1)} - \xi_s$, and the number of subdivisions increased by designating $\xi_s + \delta$ as $x^{(i_s+1)}$ and $x^{(i_s+1)}$ as $x^{(i_s+2)}$, etc. Since the inequality

$$V_n\alpha(x) - \sum_i \left| \overset{(n)}{\underset{i}{\Delta}} \alpha(x) \right| \geqq 0$$

is satisfied for the part of T where the inequality $\xi_s < x_s < \xi_s + \delta$ is not satisfied, the inequality (2) must be satisfied for the part of T where

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* Fréchet, loc. cit. (Nouvelles Annales), p. 247.

† Fréchet, loc. cit., p. 245.

‡ It is not necessary here to distinguish between Lebesgue's singular segments, and segments containing his singular points. Lebesgue, Annales Scientifiques de l'École Normale Supérieure, Ser. 3, Tome 27 (1910), p. 410.

$\xi_s \leqq x_s \leqq \xi_s + \delta$. The inequality

$$\sum_{i_1, \cdots, i_{s-1}, i_{s+1}, \cdots, i_n} \left| \overset{(n)}{\underset{i}{\Delta}} \alpha(x) \right| > \frac{k}{2} \qquad (x^{(i_s)} = \xi_s)$$

follows. If N is the number of terms in this summation, there must be at least one of them which is greater than $k/2N$, and consequently there is a set of values for $x_1, \cdots, x_{s-1}, x_{s+1}, \cdots, x_n$ such that

$$\left| \alpha(x) \left|_{x_s = \xi_s + \delta} - \alpha(x) \right|_{x_s = \xi_s} \right| > \frac{k}{2^n N}. \tag{3}$$

Since the right member of this inequality is independent of δ, $\alpha(x)$ is discontinuous in x_s at this point. To prove the condition to be sufficient it will be assumed that an inequality such as (3) is satisfied. The left member is seen to be equal to the expression

$$\left| \overset{(n)}{\underset{i}{\Delta}} \alpha(x) \right| (x^{(i_r)} = a_r,\ x^{(i_r+1)} = x_r,\ r \neq s;\ x^{(i_s)} = \xi_s,\ x^{(i_s+1)} = \xi_s + \delta),$$

and consequently the variation of $\alpha(x)$ in the region $(\xi_s \leqq x_s \leqq \xi_s + \delta)$ cannot approach zero with δ.

Since $V_n\alpha(x)$ is assumed to be finite the singular $(n-1)$-flats must be denumerable. If two functions $\alpha(x)$ and $\beta(x)$ are equal excepting on a denumerable set of $(n-1)$-flats $x_s = \xi_s^{(j)}$ $(j = 1, 2, \cdots)$, none of which are on the boundary of T, the equation

$$\int_T f(x) d_n\alpha(x) = \int_T f(x) d_n\beta(x)$$

must be satisfied, since T can be divided into as small subdivisions as is desired without making any $x^{(i_s)} = \xi_s^{(j)}$. It follows that the integral (1) is independent of the values assigned to $\alpha(x)$ at points where it is discontinuous in any argument. The function $\alpha(x)$ may then be assumed to satisfy the equations

$$\alpha(x) = \lim_{\epsilon=0} \alpha(x_1, \cdots, x_r + \epsilon, \cdots, x_n) \qquad (r = 1, 2, \cdots, n), \tag{4}$$

excepting on the boundary of T. Such a function, which also vanishes at $x_r = a_r$ and has finite variation, will be called regular in this paper. This is more convenient for our purposes than to make a regular function satisfy the equation $2\alpha(x) = \alpha(x+0) + \alpha(x-0)$, as Fréchet has done.*

§ 2. **Some properties of Stieltjes integrals.** It follows directly from the definition that the inequality

$$\left| \int_T f(x) d_n\alpha(x) \right| \leqq mfV_n\alpha(x)$$

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* Fréchet, loc. cit. (Transactions), p. 217.

is always satisfied, where mf is the maximum of $|f(x)|$. It will now be proved that if $\alpha(x)$ is regular, when an $\epsilon > 0$ is given, a continuous $f(x)$ can be found which satisfies the inequality

$$(5)\qquad \int_T f(x)d_n\alpha(x) > mf(V_n\alpha(x) - \epsilon).$$

The region T can be subdivided in such a way that the inequality

$$(6)\qquad V_n\alpha(x) - \sum_i \left|\overset{(n)}{\underset{i}{\Delta}}\alpha(x)\right| < \frac{\epsilon}{2}$$

is satisfied without using any point where $\alpha(x)$ is discontinuous as an $x^{(i_s)}$, excepting perhaps on the boundary of T. Then a δ can be chosen so small that the sum of the variations of $\alpha(x)$ in the regions

$$(7)\quad x^{(i_s)} - \delta \leqq x_s \leqq x^{(i_s)} + \delta; \qquad a_r \leqq x_r \leqq b_r, \qquad r \neq s \qquad (s = 1, 2, \cdots, n),$$

shall be less than $\epsilon/4$. The function $f(x)$ will be defined as ± 1 in each of the regions

$$x^{(i_r)} + \delta_a \leqq x_r \leqq x^{(i_r+1)} - \delta_b \qquad (r = 1, 2, \cdots, n),$$

according as $\overset{(n)}{\underset{i}{\Delta}}\alpha(x)$ is positive or negative, where $\delta_a = \delta$ excepting when $x^{(i_r)} = a_r$ when it is zero, and $\delta_b = \delta$ excepting when $x^{(i_r+1)} = b_r$ when it is zero. In the remainder of T, that is in the regions (7), $f(x)$ will be defined so as to be continuous everywhere and not greater than 1 in absolute value. Under these circumstances the inequality

$$(8)\qquad \sum_i \left|\overset{(n)}{\underset{i}{\Delta}}\alpha(x)\right| - \int_T f(x)d_n\alpha(x) < \frac{\epsilon}{2}$$

must be satisfied. Inequality (5) follows directly from inequalities (6) and (8). This completes the proof that $V_n\alpha(x)$ is the least upper bound of the expression

$$\left|\frac{1}{mf}\int_T f(x)d_n\alpha(x)\right|,$$

when $\alpha(x)$ is regular.

It follows from this theorem that if $\alpha(x)$ and $\beta(x)$ are regular and satisfy the equation

$$\int_T f(x)d_n\alpha(x) = \int_T f(x)d_n\beta(x),$$

identically in $f(x)$, they must be identical, since this equation can be put in the form

$$\int_T f(x)d_n[\alpha(x) - \beta(x)] = 0,$$

and consequently the variation of $\alpha(x) - \beta(x)$ is zero. As $\alpha(x) - \beta(x) = 0$ at $x_r = a_r$ its numerical value cannot exceed its variation.

§ 3. **Linear functionals of n-spreads.** The proof by Riesz that a linear functional of a curve can be expressed as a Stieltjes integral, which has already been referred to, can easily be extended to functionals of n-spreads. It will now be given for the case $n = 2$, and then the results stated for the more general case.

The functional $A[f(x, y)]$ will be assumed to be continuous with zeroth order,* and to be determined by the values taken by $f(x, y)$ in any region contained in T. Then $f(x, y)$ can be defined arbitrarily in the rest of T with the one condition that it be continuous. Since $A[f(x, y)]$ is linear there must be a constant M_A, independent of $f(x, y)$, such that the inequality

$$|A[f(x, y)] \leqq M_A mf$$

is satisfied. It will be assumed that M_A is the smallest such constant. As Riesz has proved for the simpler case, the field of functions for which $A[f(x, y)]$ is defined can be extended to include every $\varphi(x, y)$ which is the limit of a sequence of continuous functions which satisfy the inequalities

$$f_1(x, y) \geqq f_2(x, y) \geqq f_3(x, y) \cdots,$$

and every linear combination of such functions.† In the same way it can be proved that if there is a sequence of functions such as $\varphi(x, y)$, not necessarily continuous, which approach a limit $\psi(x, y)$ and satisfy the inequalities

$$\varphi_1(x, y) \geqq \varphi_2(x, y) \geqq \varphi_3(x, y) \cdots, \tag{9}$$

and continuous functions $f_i(x, y)$ can be found such that $f_i \geqq \varphi_i \geqq f_{i+1}$, then the equation

$$A[\psi(x, y)] = \lim_{m=\infty} A[\varphi_m(x, y)]$$

is satisfied.

A function $f(x, y; \xi, \eta)$ will be defined as identically zero when $\xi = a_1$ or $\eta = a_2$, and for other values of ξ and η as equal to 1 in the region $(a_1 \leqq x \leqq \xi;\ a_2 \leqq y \leqq \eta)$ and equal to 0 elsewhere. It is easily proved to be the limit of a decreasing sequence of continuous functions.‡ The function $\alpha(x, y)$ can then be defined by the equation

$$\alpha(\xi, \eta) = A[f(x, y; \xi, \eta)].$$

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* If the continuity is of a higher order see Fischer, Bulletin of the American Mathematical Society, vol. 23 (1916), p. 88.

† Riesz, loc. cit., pp. 10–13.

‡ Compare with Riesz, loc. cit., p. 13.

It follows from the definition of $f(x, y; \xi, \eta)$ that $\alpha(a_1, y) = \alpha(x, a_2) = 0$. To prove that it is regular, a sequence of decreasing constants $\epsilon_1, \epsilon_2, \cdots$ will be chosen approaching zero as a limit. Then the sequences $A[f(x, y; \xi + \epsilon_m, \eta)]$ and $A[f(x, y; \xi, \eta + \epsilon_m)]$ both approach $\alpha(\xi, \eta)$, and the inequalities analogous to (9) are satisfied. It follows directly that the equations analogous to (4) are satisfied and $\alpha(x, y)$ is regular. By definition $\underset{i,j}{\Delta} f(x, y; \xi, \eta) = 1$ in the region

$$(10) \qquad T_{ij}: \quad \xi_i < x \leqq \xi_{i+1}; \qquad \eta_j < y \leqq \eta_{j+1},$$

and vanishes everywhere else. If $i = 0$ the first of these inequalities should read $a_1 \leqq x \leqq \xi_1$, and similarly if $j = 0$. The inequality

$$\sum_{i,j} | \underset{ij}{\Delta} \alpha(\xi, \eta) | = A[\sum_{i,j} \pm \underset{i,j}{\Delta} f(x, y; \xi, \eta)] \leqq M_A$$

follows. Thus the variation of $\alpha(x, y)$ cannot be greater than M_A.

It will now be proved that the equation

$$(11) \qquad A[f(x, y)] = \int_T f(x, y) d_2 \alpha(x, y)$$

is satisfied. The region T will be divided arbitrarily into the set of rectangles T_{ij}, and an arbitrary point of T_{ij} designated by (ξ_i', η_j'). A function $\varphi(x, y)$ will then be defined by the equation

$$\varphi(x, y) = \sum_{i,j} f(\xi_i', \eta_j') \underset{i,j}{\Delta} f(x, y; \xi, \eta).$$

It follows that

$$A[\varphi(x, y)] = \sum_{i,j} f(\xi_i', \eta_j') \underset{i,j}{\Delta} \alpha(\xi, \eta).$$

The definition of $\varphi(x, y)$ implies that it is equal to $f(\xi_i', \eta_j')$ in T_{ij}. Then if ω is the greatest oscillation of $f(x, y)$ in any T_{ij}, the inequality

$$| A[f(x, y)] - \sum_{i,j} f(\xi_i', \eta_j') \underset{i,j}{\Delta} \alpha(\xi, \eta) | = | A[f - \varphi] | \leqq \omega M_A$$

must be satisfied, and equation (11) follows.

The proof for the general case is essentially the same as the above and need not be repeated here. The results of the last two sections may be stated in the following way. *If a linear functional $A[f(x)]$, which is continuous with zeroth order, is determined by the values taken by $f(x_1, \cdots, x_n)$ in the region T, then there is a unique regular function $\alpha(x_1, \cdots, x_n)$, independent of f, which satisfies the equation*

$$A[f(x)] = \int_T f(x) d_n \alpha(x),$$

and the variation of α *is the least upper bound of the absolute value of* $A[f(x)]/mf$.

As this same property of functionals of curves is the foundation of Fréchet's work on bilinear functionals,[*] his proof that a bilinear functional can be expressed as a double Stieltjes integral can also be extended to functionals of n-spreads.

COLUMBIA UNIVERSITY.

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[*] Fréchet, loc. cit. (Transactions).

CORRECTION.

BY L. I. HEWES.

On p. 197, Vol. XVIII, the equation in Fig. 1 should be

$$Z_1Z_2 - Z_3 + \sqrt{1 + Z_2^2}\sqrt{1 + Z_1^2} = 0$$

and Z_1 in the curve net should read Z_3.